FOURIER COEFFICIENTS OF LONG-RANGE BEAM-BEAM HAMILTONIAN VIA TWO-DIMENSIONAL BESSEL FUNCTIONS∗

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INTRODUCTION

The Fourier-expansion coefficients of the accelerator Hamiltonian appear naturally in analytical calculations of amplitude detuning, low-order normal form and related to it driving terms. In studies of large-range beam-beam interaction with neglected bunch-length effects, i.e. a two-dimensional Hamiltonian \( H(x,y) \), the coefficients \( c_{mk} \) are usually expressed through modified Bessel functions of the first kind \( I_n(u) \) [1–3], or their relatives [4], and given as single integrals over sum of the products of two \( I_n(u) \). High \( \sim 40 \) orders \( n \) are needed in the following representation. The method is applied to the nominal scenario HL-LHC lattice and benchmarked against MadX simulations of detuning.

HAMILTONIAN COEFFICIENTS VIA TWO-ARGUMENT BESSEL FUNCTIONS

When written in terms of unperturbed action-angle variables, the Hamiltonian \( H(x,y) \) describing the beam-beam kick at a head-on (HO), or long-range (LR) interaction point (IP) depends on \( \alpha_z, \beta_z (z=x,y) \) – normalized test-particle amplitude and full separation at this IP. We assume round-beam optics and equal emittances of weak and strong beam, but possibly “flat-beam” long-range IP, i.e. one with \( \beta_x \neq \beta_y \), so that \( r \equiv \frac{\alpha_w}{\alpha_q} \neq 1 \). In this latter case, let us use the symmetry of Interaction Region 5 (IR5), where the beams are separated in \( x \) direction. Here weak and strong-beam sigmas are related by \( \sigma^w_x = \sigma_w, \sigma^w_y = \sigma_q, \sigma^y_x = x = r\sigma_x a_x \sin \phi_x, y = \sigma_w \sin \phi_y \). Thus, for \( r \neq 1 \), the formulae below are valid for IR5, while IR1 (vertical separation) can be treated symmetrically. The case \( r = 1 \) is generic (the formulae are valid for any insertion).

For an IP in IR5 the Hamiltonian, in units of \( \frac{N_b \sigma_w}{\beta_w} \), is:

\[
H(x,y) = \int_0^1 \frac{dt}{t g(t)} \frac{1}{1 - e^{-r(P_\perp + P_z)}};
\]

\[
P_\perp = \frac{1}{2} (\bar{d}_x + \bar{z}_q \sin \phi_x)^2,
\]

\[
\bar{d}_x = r \alpha_x, \quad \bar{d}_y = d_y, \quad \bar{\phi}_x = \alpha_x, \quad \bar{\phi}_y = \frac{r \alpha_y}{g(t)}.
\]

where \( \gamma \) is the relativistic factor, \( N_b \) is the bunch population and \( g(t) \equiv \sqrt{1 + (r^2 - 1) t} \). Removing the bar in all variables gives the generic case of round-beam IP \( (r = 1, g = 1) \). By expanding \( P_z \), we have the relations:

\[
-t P_z = -u_1^{(z)} \sin \phi_x + 2u_2^{(z)} \sin^2 \phi_x + u_3^{(z)} \quad (2)
\]

\[
u_1^{(z)} = t \bar{d}_x, \quad u_2^{(z)} = -\frac{t}{4} \bar{z}_q, \quad u_3^{(z)} = -\frac{t}{2} \bar{z}_q, \quad u_4^{(z)} = \bar{u}_2^{(z)} + u_3^{(z)} = -\frac{t}{2} \bar{d}_x - \bar{u}_2^{(z)} - u_4^{(z)} - u_2^{(z)}.
\]

Using Eqn. (2), the Fourier coefficients

\[
c_{mk} = \frac{1}{4 \pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-im\phi_x - ink\phi_y} d\phi_x d\phi_y
\]

are expressed [1], [2], [7] as integrals over Bessel series (the Introduction). Somewhat more directly, let us combine (2) with the generating function for two-argument Bessel functions:

\[
e^{-u_1 \sin \phi_x + u_2 (2 \sin \phi_x^2 - 1)} = \sum_{k=0}^{\infty} i^k I_k(u_1, u_2) e^{ik\phi_x}.
\]

The result is here \( \delta = 1 \) if \( m = k = 1 \) and 0 otherwise:

\[
c_{mk} = \int_0^1 \frac{dt}{t g(t)} [\delta - \bar{Q}_m^{(x)}(t) \bar{Q}_k^{(y)}(t)]
\]

where \( \bar{Q}_m^{(x)}(t) \) and \( \bar{Q}_m^{(y)}(t) \) are given by the closed expressions.

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The $Q$s can be written either in terms of $I$, or $\Lambda$:

$$Q_{m}^{(z)}(t) \equiv i^{m} e^{-i\bar{z}} I_{m}(u_{1}^{(z)}, u_{2}^{(z)}) = i^{m} e^{-i\bar{z}} I_{m}(u_{1}^{(z)}, u_{2}^{(z)}) A_{m}(u_{1}^{(z)}, u_{2}^{(z)}).$$  

(6)

Barring small differences in notation, and the fact that for $r \neq 1$ the IRS symmetry has already been embedded, this is identical to [2], see e.g. Eqn 52. Notice that the first form (5) contains no barred variables while the second (6) does, but is more intuitive: for in-plane LR collision the exponent factor is just the squared distance $a_{s} - |d_{s}|$ between the weak-beam particle amplitude and the strong-beam centroid.

Particular cases follow directly. E.g. for in-plane LR collision one uses that in the $y$-plane $I_{0}(0, 0) = 1$, to get: $c_{m} \equiv c_{m,0} = \int_{0}^{1} dt \left( \delta(m) - K(t) \right)$, where $K(t) = i^{m} e^{-\frac{i}{2}a_{s}^{2}} e^{-\frac{iz}{2}} I_{0}(t a_{d}s, -\frac{i}{2}a_{s}^{2})$.

**RECURSIVE PROPERTIES**

The functions $I_{m}(u_{1}, u_{2})$ obey one recursion

$$u_{1} \left[ I_{m-1} - I_{m+1} \right] + 2u_{2} \left[ I_{m-2} - I_{m+2} \right] = 2m I_{m},$$  

(7)

and two derivative properties:

$$\frac{\partial I_{m}}{\partial u_{1}} = \frac{1}{2} \left[ I_{m-1} + I_{m+1} \right]; \quad \frac{\partial I_{m}}{\partial u_{2}} = \frac{1}{2} \left[ I_{m-2} + I_{m+2} \right]$$  

(8)

(and a similar one for $\Lambda$). On the other hand, by rewriting (4) using (5) the coefficients $c_{mk}$ are:

$$\int_{0}^{1} \frac{dt}{tg(t)} \left[ \delta - i^{m+k} e^{i\bar{z}d_{s}} I_{m}(u_{1}^{(z)}, u_{2}^{(z)}) I_{k}(u_{1}^{(z)}, u_{2}^{(z)}) \right]$$  

(9)

Hence the higher-order resonance coefficients are not independent. For fixed arguments $u_{1}$ and $u_{2}$, (7) allows to find easily $I_{m}$ for all $m$, given the first four ($I_{m}$ for $m = 0, 1, 2, 3$) – useful also in numerical calculations (see below). A recursive procedure for $c_{mk}$ has been found – albeit rather difficult to solve, so not used in numerical calculations – where the complication arising from $I_{m}$ being under an integral sign is compensated by the additional differential relations (8).

Our (preliminary) conclusion is that at least in principle, the $c_{mk}$ can all be expressed through the ones of order up to and including order $m = 3$ ("octupole"), assuming a beam-beam "multipole" has been defined in the new resonance basis in a way similar to "usual" multipoles.

The following conjecture is then made. If, as a result of lumped correction, local, i.e. at this IP, compensation of all terms to order $m = 3$ has taken place, then all resonance terms are also canceled. On the other hand, as we will see, (8) allows to express amplitude dependent tune-shifts using the first three ($I_{m}$ for $m = 0, 1, 2$). To summarize, if terms up to order 2 ("sextupole") are locally minimized, then the footprint is reduced. If in addition the $m = 3$ term is minimized, then this leads to all resonance terms being small. Same or similar conclusions have been made in [8].

**FOOTPRINT**

The nonlinear detunings with amplitude $\Delta Q_{x}$ given by the partial derivatives of $c_{00}$ over the actions $J_{x}, J_{y}$.

By setting $m = k = 0$ in (4) and replacing $\delta$ with unity:

$$c_{00} = \int_{0}^{1} \frac{dt}{tg(t)} \left[ 1 - Q_{0}^{(x)}(t)Q_{0}^{(y)}(t) \right];$$  

$$\frac{\partial c_{00}}{\partial \alpha_{x}} = -\int_{0}^{1} \frac{dt}{tg(t)} \frac{\partial Q_{0}^{(y)}(t)}{\partial \alpha_{x}} Q_{0}^{(y)}(t)$$  

(10)

(and similar for $y$); 

$$\Delta Q_{x} = 2\xi \frac{1}{\alpha_{x}} \frac{\partial c_{00}}{\partial \alpha_{x}}, \quad \text{where} \quad z = x \ or \ y. \quad (11)$$

Here $\xi \equiv \frac{N_{b}m}{4\pi\gamma}$ is the beam-beam parameter (both $H$ and $c_{00}$ are in units of $\frac{N_{b}m}{4\pi\gamma}$) and we have used $\frac{da_{x}}{dt} = -\frac{1}{\epsilon_{a_{x}}}$. According to (11) one needs to compute (10) twice, where under the integral there is the product of $Q$ and a derivative of $Q$ (with $x \leftrightarrow y$). The derivative can be taken using a property as (8). The result, in terms of $\Lambda$, is:

$$Q_{0}^{(z)} = e^{-\frac{i}{2}(d_{s} - \bar{z})^{2}} \Lambda_{0};$$  

$$\frac{\partial Q_{0}^{(z)}}{\partial \alpha_{z}} = \eta_{x} e^{-\frac{i}{2}(d_{s} - \bar{z})^{2}} \left[ -\frac{i}{2} \left( \Lambda_{0} + \Lambda_{2} \right) + \bar{z} \Lambda_{1} \right];$$  

$$\eta_{x} \equiv \xi t, \quad \eta_{y} \equiv \xi g(t);$$  

$$\Lambda_{0,1,2} \equiv \Lambda_{0,1,2}(u_{1}^{(z)}, u_{2}^{(z)}).$$  

(12)

As advertised, the footprint depends on the first three $\Lambda$. Finally notice that in (10), since $1/t$ cancels and $g > 0$ for any $r$, there is no singularity under the integral.

Familiar expression for single-plane head-on, IP without offset, follow from $I_{m-2q}(0) = \delta(m - 2q)$, or alternatively from $I_{m}(0, u_{2}) = I_{m/2}(u_{2})$ (only even $m$ remain).

The tune-shift expressions derived in [3] ($r = 1$ only) follow from (10) by replacing in it $\Lambda$ with its generating function form (3), reversing the order of integration and using the fact that for $r = 1$ (only!) the integral over $t$ is solvable.

**NUMERICAL IMPLEMENTATION AND COMPARISON WITH MADX**

For numerical calculations of both $c_{ml}$ and detuning we have encoded the two-dimensional Bessel functions either in Mathematica (any Bessel arguments), or as a fortran code (faster). In fortran, we take advantage of the recursion property: since all $c_{mk}$ depend on only four two-argument functions $I_{mk}$, these are precomputed and stored as four tables. Thus only cases $n = 0, 1, 2, 3$ need be computed to high accuracy as the rest of $I_{m}$ follow recursively. The integral over $t$ is taken using bi-linear approximation of these tables.

In what follows, our goal is to verify the Hamiltonian by comparing expressions (11), (12) with MadX tracking (dynamaplot) and prove overall applicability of the method at nominal HL-LHC settings, i.e. large Bessel arguments.
summing range \( q_{\text{max}} \) and size of the prestored fortran arrays.

Figure 1: \( a_z \) ranges – 15 angles (left) and 3 angles (right).

<table>
<thead>
<tr>
<th>dx</th>
<th>dy</th>
<th>( \alpha_x ) [mm]</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0.087</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2: Sample IPs with \( r = 1 \): 1) head-on IP1 and 5; 2) single long range IP closest to IP5, bbip5L1 for half the nominal crossing angle (required \( q_{\text{max}}=25 \); 3) same as 2), but full nominal crossing angle (required \( q_{\text{max}}=35 \))

<table>
<thead>
<tr>
<th>dx</th>
<th>dy</th>
<th>( \alpha_x ) [mm]</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.236</td>
<td>0.176</td>
<td>1.0</td>
<td></td>
</tr>
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</table>

Figure 4: IP with \( r=2 \): nominal HL-LHC (bbip5L10)

The nominal HL-LHC setup implies very small tune-shifts per long-range IP, hence the need of high accuracy – the relative difference (\( \Delta Q/Q \)) between tracking and analytic formula is predominantly better than \( 4 \times 10^{-4} \). For each plot, before the comparison is made, tiny tune shifts \( \sim 5 \times 10^{-5} \) still present in the beam-beam free lattice are subtracted from the MadX output. For case 3 on Fig. 2 the \( q_{\text{max}} \) had to be increased from 25 to 35 – otherwise the last two red circles at the bottom would deviate substantially. Largest Bessel arguments correspond to the setup in Fig. 4: maximum \( a_z = 12 \), \( d_x \sim 19 \) with \( r=2 \). In this last case the Mathematica’s computing time was \( \sim 300 \) sec.

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REFERENCES