THEORETICAL FORMULATION OF IMPROVED SASE FEL BASED ON SLIPPAGE ENHANCEMENT SCHEME∗

C.-Y. Tsai1†, J. Wu1‡, C. Yang1,2, M. Yoon3, and G. Zhou1,4
1SLAC National Accelerator Laboratory, Menlo Park, California, USA
2NSRL, University of Science and Technology of China, Hefei, Anhui, China
3Department of Physics, Pohang University of Science and Technology, Pohang, Korea
4Institute of High Energy Physics, and UCAS, Chinese Academy of Sciences, Beijing, China

Abstract
A method to improve the spectral brightness of self-amplified spontaneous emission (SASE) based on slippage enhancement has been proposed [1–4]. An implementation is to insert a series of magnetic chicanes to introduce a path-length delay of the electron beam to the radiation beam. By correlating the electron slices of neighboring cooperation distances this can lengthen the collective interaction and thus enhance the spectral brightness. In the existing literature most studies rely on numerical simulations and there is limited work on analytical analysis. In this paper we formulate the problem of slippage enhanced SASE (SeSASE) high-gain FEL with inclusion of by-pass magnetic chicanes. The analysis takes the finite energy spread of the electron beam and the nonzero momentum compaction of the chicanes into consideration. The evolution of spectral bandwidth of SeSASE is compared with that of usual SASE in theory. The effects of finite beam energy spread and non-isochronism are also quantified.

THEORETICAL FORMULATION
In the theoretical formulation we largely follow the notation used in the book by K.-J. Kim et al. [5]. Let us start from the single-particle equations of motion

\[ \frac{d\theta_j}{dz} = 2k_u \eta_j \]  

(1)

\[ \frac{d\eta_j}{dz} = \chi_1 E(\theta_j) + c.c. = \chi_1 \int dv E_v(z)e^{i\nu\eta} + c.c. \]  

(2)

where \( \theta_j \) is the ponderomotive phase of \( j \)-th particle, \( \eta_j = (\gamma_j - \gamma_R)/\gamma_R \) is the energy deviation to the resonance one \( \gamma_R \). \( E \) and \( E_v \) are the electric field in time and normalized frequency domain \( \nu = \omega/\omega_0 \), respectively, \( k_1 = eK[J]/4\epsilon_0 \gamma_R \) and \( \chi_1 = eK[J]/2y^2mc^2 \), with \( K \) the dimensionless undulator parameter and \( J \) the coupling factor. The 1-D wave equation based on slowly varying envelope approximation can be formulated as

\[ \left( \frac{\partial}{\partial z} + k_u \frac{\partial}{\partial \theta} \right) E(\theta) = -\kappa_1 n_e \frac{2\pi}{N_1} \sum_{j=1}^{N_e} e^{-i\theta_j(z)} \delta \left[ \theta - \theta_j(z) \right] \]  

(3)

where \( N_1 \) is the number of electrons in one radiation wavelength \( \lambda_1 \). The electron phase space distribution can be described using the Klimontovich distribution function to retain the discrete nature of the electrons, \( F(\theta, \eta; z) = \frac{k_1}{\rho \epsilon_0} \sum_{j=1}^{N_1} \delta \left[ \theta - \theta_j(z) \right] \delta \left[ \eta - \eta_j(z) \right] \) in which the dynamics is governed by the continuity equation

\[ \frac{dF}{dz} = \frac{\partial E}{\partial \theta} + \frac{\partial F}{\partial \eta} \frac{dE}{dz} + \frac{\partial F}{\partial \eta} \frac{dE}{dz} = 0 \]  

in general the continuity equation is nonlinear since \( d\eta/dz \) depends on \( F \) as well. In the following analysis, we are interested in the linear regime where the electron phase space distribution can be well separated into the smooth background and the small perturbing part, in which the information of shot noise and perturbation due to FEL process is contained. In addition we make the coasting beam approximation in describing the electron beam distribution. Under this approximation we have neglected the situation when the radiation field slips over the edge of an electron bunch, i.e., the slippage-induced superradiance FEL [6] is excluded in our analysis. After linearizing the continuity equation and transforming to the normalized frequency domain, we obtain

\[ \left( \frac{\partial}{\partial z} + 2i\nu k_u \eta \right) F_v(\eta; z) + \chi_1 E_v(z) \frac{dV}{d\eta} = 0 \]  

(4)

where \( V(\eta) \) the electron beam energy distribution. The wave equation Eq. (3) represented in the frequency domain is

\[ \left( \frac{\partial}{\partial z} + i\Delta k_u \right) E_v(z) = -\kappa_1 n_e \int d\eta F_v(\eta; z) \]  

(5)

Here we note that Eqs. (4) and (5) are more general and can be reduced to those based on Bonifacio et al. collective-variable description in the cold-beam limit [7]. To solve Eqs. (4) and (5) as an initial value problem, we shall employ Laplace transform and the resultant electric field in the frequency domain can be expressed as

\[ E_v(z) = \int e^{-i\nu(\rho + \mu)z} 2\pi i D(\mu) \left[ E_v(0) + \frac{ik_z n_e}{2pku} \sum_{j=1}^{N_1} e^{-i\theta_j(\nu)} \sum_{\nu=1}^{N_e} \delta \left[ \theta - \theta_j(z) \right] \right] d\mu \]  

(6)

where the dispersion relation \( D(\mu) = \mu - \frac{N_1}{\sqrt{\lambda_1}} - \int \frac{V(\nu)d\nu}{\nu\rho^2} = 0 \). \( \rho \) is the FEL or Pierce parameter. The contour integral should enclose all the singularities and draw in the lower complex-\( \mu \) plane. The electron phase space distribution can be obtained \( F_v(\nu; \rho, \mu) \), provided \( E_v(\nu; \rho, \mu) \) is given.

02 Photon Sources and Electron Accelerators

A06 Free Electron Lasers
Now we consider the SeSASE FEL process with a schematic layout shown in Fig. 1. Hereafter we assume the chicane(s) shall be placed where the FEL process is dominated by the unstable root, say $\mu_3$ of $D(\mu)$. This situation corresponds to that occurs after about two FEL gain lengths. In what follows we aim to derive the matrix representation for transport of $E_f$ and $F_f$ through undulator section and magnetic chicane. The electric field at the exit of $n$-th undulator section can be expressed as

$$E_f(z_f^{(n)}) = e^{i\Delta f_{\mu_3}}[E_f(z_f^{(n-1)}) + \int d\eta F_f(\eta) e^{i\Delta f_{\mu_3}}]$$

and the corresponding electron phase space distribution $F_f(\eta, z_f^{(n)}) = \frac{ik_e}{\beta_c \gamma_c z_f^{(n)}} E_f(z_f^{(n)})$. After the chicane, the electron beam is bypassed and the electric field acquires an additional phase. The resultant electric field and the electron phase space distribution become

$$E_f(z_f^{(n+1)}) = e^{i\Delta f_{\mu_3}}[E_f(z_f^{(n)}) + \int d\eta F_f(\eta) e^{i\Delta f_{\mu_3}}]$$

and

$$F_f(\eta, z_f^{(n+1)}) = \frac{ik_e}{\beta_c \gamma_c z_f^{(n+1)}} e^{i\Delta f_{\mu_3} + \Delta(\mu_3)}.$$
Now we consider the simplest SeSASE case, i.e., \( N = 2 \). Then we have \( E_{\text{SeSASE}}^{v,f} = (e^{i\Delta \nu \phi} - \rho H) E_{\text{SASE}}^{v,f} \). The power spectral density can be expressed as 
\[
\frac{dP}{d\nu} E_{\text{SeSASE}}^{v,f} = S(\phi, H) \frac{dP}{d\nu} E_{\text{SASE}}^{v,f}
\]
where \( S(\phi, H) = 1 + \rho^2 |H|^2 - 2 \rho |H| \cos(\Delta \nu \phi + \phi) \). Here we note that the shape function \( S \) is general in the sense that it contains the finite energy spread of the electron beam and allows the nonzero \( R_{56} \). The spectrum bandwidth can then be evaluated analytically by 
\[
\sigma_{\nu} = \frac{1}{\nu_\text{m} \sigma_{\nu 0}} \sqrt{\frac{1 + \rho^2 |H|^2 - 2 \rho |H| \cos(\Delta \nu \phi + \phi)}{1 + \rho^2 |H|^2 - 2 \rho |H| \cos(\Delta \nu \phi + \phi) e^{-\nu_\text{m} \sigma_{\nu 0}^2/2}}}
\]
Figure 3 shows the shape function as a function of \( R_{56} \) for several different frequency detunes and the spectrum bandwidth of SeSASE as a function of phase shift \( \phi \) for \( N = 2 \) case. In the numerical illustration we assume \( \nu_\text{m} = 0.31 \) nm, \( \rho = 5.6 \times 10^{-4} \), the full-width (uniform) energy spread \( 5 \times 10^{-4} \), and the 1-D gain length \( L_G = 2.13 \) m.

Similarly, for the case of \( N = 3 \), we have \( E_{\text{SeSASE}}^{v,f} = (e^{i\Delta \nu \phi_1} - \rho H(3)) (e^{i\Delta \nu \phi_2} - \rho H(2)) E_{\text{SASE}}^{v,f} \) and the power spectral density \( \frac{dP}{d\nu} E_{\text{SeSASE}}^{v,f} = S(\phi_2, \phi_3, H_2, H_3) \frac{dP}{d\nu} E_{\text{SASE}}^{v,f} \). The spectrum bandwidth can also be obtained analytically.

In the most general case, the shape function can be symbolically formulated as 
\[
E_{\text{SeSASE}}^{v,f} = \frac{1}{\nu_\text{m} \sigma_{\nu 0} \sqrt{2\pi}} \int d\nu \rho |H| \cos(\Delta \nu \phi) e^{-\nu_\text{m} \sigma_{\nu 0}^2/2} \text{e}^{i\Delta \nu \phi}
\]

As a result, the shape function \( S \) is given by
\[
S(\phi, H) = 1 + \rho^2 |H|^2 - 2 \rho |H| \cos(\Delta \nu \phi + \phi) \]

Figure 3: (Left) The shape function \( S \) as a function of momentum compaction \( R_{56} \) for different \( \Delta \nu \). (Right) The relative frequency bandwidth as a function of phase shift \( \phi \) for different \( R_{56} \) at \( \nu = 10 L_G \).

\section*{Summary and Outlook}

In this paper we have formulated the slippage enhanced SASE (SeSASE) high-gain FEL process with inclusion of by-pass magnetic chicanes and extension to a general \( N \) undulator-chicane moduli. The analysis takes the finite energy spread of the electron beam and the nonzero momentum compaction of the chicane into consideration. The evolution of spectral bandwidth of SeSASE is derived and expressed in a combined form with that of usual SASE case. The effects of finite beam energy spread and non-isochronisity are also explicitly expressed. These analytical expressions may be used to improve or optimize the SeSASE performance. Further investigation on improving the SeSASE performance with varying bunch current density is ongoing.

\section*{References}

[1] J. Wu et al., “Generation of Longitudinally Coherent Ultra High Power X-Ray FEL Pulses by Phase and Amplitude Mix-


